Stabbing line segments with disks: complexity and approximation algorithms ⋆

Konstantin Kobylkin

Institute of Mathematics and Mechanics, Ural Branch of RAS,
Sophya Kovalevskaya str. 16, 620990 Ekaterinburg, Russia,
Ural Federal University, Mira str. 19, 620002 Ekaterinburg, Russia,
kobylkinks@gmail.com

Abstract. Computational complexity and approximation algorithms are reported for a problem of stabbing a set of straight line segments with the least cardinality set of disks of fixed radii $r > 0$ where the set of segments forms a straight line drawing $G = (V, E)$ of a planar graph without edge crossings. Close geometric problems arise in network security applications. We give strong NP-hardness of the problem for edge sets of Delaunay triangulations, Gabriel graphs and other subgraphs (which are often used in network design) for $r \in [d_{\min}, \eta d_{\max}]$ and some constant η where $d_{\max}$ and $d_{\min}$ are Euclidean lengths of the longest and shortest graph edges respectively. Fast $O(|E| \log |E|)$-time $O(1)$-approximation algorithm is proposed within the class of straight line drawings of planar graphs for which the inequality $r \geq \eta d_{\max}$ holds uniformly for some constant $\eta > 0$, i.e. when lengths of edges of G are uniformly bounded from above by some linear function of r.

Keywords: computational complexity, approximation algorithms, Hitting Set, Continuous Disk Cover, Delaunay triangulations

1 Introduction

Numerous applications from security, sensor placement and robotics lead to computational geometry problems in which one needs to find the smallest cardinality set C of points on the plane having bounded (in some sense) visibility area such that each piece of the boundary of a given geometric object or any part of the complex (i.e. set of edges or faces) of a plane graph is within visibility area of some point from C, see e.g. [5], [12]. Refining complexity statuses and designing approximation algorithms for these problems is still an area of active research. In this paper complexity and approximability are studied of the following problem. Intersecting Plane Graph with Disks (IPGD): given a straight line drawing (or a plane graph) $G = (V, E)$ of an arbitrary simple[1] planar graph without edge crossings and a constant $r > 0$, find the smallest cardinality set $C \subset \mathbb{R}^2$ of points (disk centers) such that each edge $e \in E$ is within Euclidean distance r

⋆ This work was supported by Russian Science Foundation, project 14-11-00109.

[1] a graph without loops and parallel edges

from some point $c = c(e) \in C$ or, equivalently, the disk of radius r centered at c intersects e.

The IPGD abbreviation is used throughout our paper to denote the above problem for simplicity of presentation.

Applications of complexity and algorithmic analysis of the IPGD problem come from network security. More specifically, IPGD represents the following model in which we are to evaluate vulnerability of some physical network to simultaneous technical failures caused by natural (e.g. floods, fire, electromagnetic pulses) and human sources. In this model network nodes are modeled by points on the plane while its physical links are given in the form of straight line segments. A catastrophic event (threat) is usually localized in a particular geographical area and modeled by a disk of some fixed radius $r > 0$. A threat impacts a network link when the corresponding disk and segment intersect. Evaluation of the network vulnerability can be posed in the form of finding the minimum number of threats along with their positions that cause all network links to be broken. Thus, it brings us to the IPGD problem assuming that network links are geographically non-overlapping. A similar setting is considered in [1] with the fixed number of threats. Furthermore, in [12] a close geometric problem is considered called the Art Gallery problem where point coverage area is affected by boundaries of its neighbouring geometric objects whereas point has circular visibility area in the case of the IPGD problem.

In this paper computational complexity and approximability of IPGD are studied for simple plane graphs with either $r \in [d_{\min}, d_{\max}]$ or $r = \Omega(d_{\max})$ where $d_{\max}$ and $d_{\min}$ are Euclidean lengths of the longest and shortest edges of G. Our emphasis is on those classes of simple plane graphs that are defined by some distance function, namely, on Delaunay triangulations, some of their connected subgraphs, e.g. for Gabriel graphs. These graphs are often called *proximity* graphs. Delaunay triangulations are plane graphs which admit efficient geometric routing algorithms [4], thus, representing convenient network topologies. Gabriel graphs arise in modeling wireless networks [14].

1.1 Related work

IPGD is related to several well-known combinatorial optimization problems. First, we have the Continuous Disk Cover (CDC) problem for the case of IPGD where G consists of isolated vertices, i.e. when segments from E are all of zero length. Strong NP-hardness is well known for CDC [9]. Second, IPGD coincides with known Vertex Cover problem for $r = 0$. Third, it is the special case of the geometric Hitting Set problem on the plane.

Hitting Set: given a family $\mathcal{N}$ of sets on the plane and a set $U \subseteq \mathbb{R}^2$, find the smallest cardinality set $H \subseteq U$ such that $N \cap H \neq \varnothing$ for every $N \in \mathcal{N}$.

IPGD coincides with Hitting Set if we set $\mathcal{N} := \mathcal{N}_r(E) = \{N_r(e)\}_{e \in E}$ and $U := \mathbb{R}^2$ where $N_r(e) = B_r(0) + e = \{x + y : x \in B_r(0),\, y \in e\}$ is Euclidean r-neighbourhood of e having form of Minkowski sum and $B_r(x)$ is the disk of radius

r centered at $x \in \mathbb{R}^2$. An *aspect ratio* of a closed convex set N with $\operatorname{int} N \neq \varnothing$[2] coincides with the ratio of the minimum radius of the disk which contains N to the maximum radius of the disk which is contained in N. For example, each set $N_r(e)$ (also called by object in the sequel) has aspect ratio equal to $1+\frac{d(e)}{2r}$ where $d(e)$ is Euclidean length of edge $e \in E$. APX-hardness of the discrete[3] Hitting Set problem is presented for families of axis-parallel rectangles, generally, with unbounded aspect ratio, [6], and for families of triangles of bounded aspect ratio [13].

1.2 Results

Our results report complexity and approximation algorithms for the IPGD problem within several classes of plane graphs under different assumptions on r. Let S be a set of n points in general position on the plane no four of which are cocircular. We call a plane graph $G = (S, E)$ a *Delaunay triangulation* if $[u, v] \in E$ iff there is a disk T such that $u, v \in \operatorname{bd} T$[4] and $S \cap \operatorname{int} T = \varnothing$. Finally, a plane graph $G = (S, E)$ is named a *nearest neighbour* graph when $[u, v] \in E$ iff either u or v is the nearest Euclidean neighbour for v or u respectively.

Hardness results. Our first result claims strong NP-hardness of IPGD within the class of Delaunay triangulations and some known classes of their connected subgraphs (Gabriel and relative neighbourhood graphs) for $r \in [d_{\min}, d_{\max}]$ and $\mu = \frac{d_{\max}}{d_{\min}} = O(|S|)$. IPGD remains strongly NP-hard within the class of nearest neighbour graphs for $r \in [d_{\max}, \eta d_{\max}]$ with a large constant η and $\mu \leq 4$. Furthermore, we have the same NP-hardness results under the same restrictions on r and μ even if we are bound to choose points of C close to vertices of G. The upper bound on μ for Delaunay triangulations is comparable with the lower bound $\mu = \Omega\left(\sqrt[3]{n^2}\right)$ which holds true with positive probability for Delaunay triangulations produced by n random independent points on the unit disk [2]. Thus, declared restrictions on r and μ define natural instances of IPGD.

An upper bound on μ implies an upper bound on the ratio of the largest and smallest aspect ratio of objects from $\mathcal{N}_r(E)$. The Hitting Set problem is generally easier when sets from $\mathcal{N}$ have almost equal aspect ratio bounded from above by some constant. Our result for the class of nearest neighbour graphs gives the problem NP-hardness in the case where objects of $\mathcal{N}_r(E)$ have almost equal constant aspect ratio.

In distinction to known results for the Hitting Set problem mentioned above our study is mostly for its continuous setting with the structured system $\mathcal{N}_r(E)$ formed by an edge set of a specific plane graph; each set from $\mathcal{N}_r(E)$ is of the special form of Minkowski sum of some graph edge and radius r disk. Our proofs are elaborate complexity reductions from the CDC problem which is intimately related to IPGD.

[2] $\operatorname{int} N$ is the set of interior points of N
[3] when U coincides with some prescribed finite set
[4] $\operatorname{bd} T$ denotes the set of boundary points of T

Positive results. Let $R(E)$ be the smallest radius of the disk that intersects all segments from the edge set E. As opposed to the cases where either $r \in [d_{\min}, d_{\max}]$ or $r \in [d_{\max}, \eta d_{\max}]$, IPGD is solvable within the class of simple plane graphs, for which the inequality $r \geq \eta R(E)$ holds uniformly for some fixed $\eta > 0$, in $O\left(k^2|E|^{2k+1}\right)$ time with $k = \left\lceil \frac{\sqrt{2}}{\eta} \right\rceil^2$. Above inequality implies an upper bound k on its optimum. Taking proof of $W[1]$-hardness into account of parameterized version of CDC [10] as well as the reduction used to prove the theorem 2 of this paper, it seems unlikely to improve this time bound to $O(f(k)|E|^c)$ for any computable function f and any constant $c > 0$.

Finally, we present an $8p(1+2\lambda)$-approximation $O(|E|\log|E|)$-time algorithm for IPGD when the inequality $r \geq \frac{d_{\max}}{2\lambda}$ holds true uniformly within a class of simple plane graphs for a constant $\lambda > 0$, where $p(x)$ is the smallest number of unit disks needed to cover any disk of radius $x > 1$. It corresponds to the case where segments from E have their lengths uniformly bounded from above by some linear function of r, or, in other words, when objects from $\mathcal{N}_r(E)$ have their aspect ratio bounded from above by $1+\lambda$. A similar but more complex $O(|E|^{1+\varepsilon})$-time constant factor approximation algorithm is given in [7] to approximate the Hitting Set problem for sets of objects whose aspect ratio is bounded from above by some constant.

2 NP-hardness results

We give complexity analysis for the IPGD problem by considering its setting where $r \in [d_{\min}, d_{\max}]$. Under this restriction on r IPGD coincides neither with known Vertex Cover problem nor with CDC. In fact it is equivalent (see the Introduction) to the geometric Hitting Set problem for the set $\mathcal{N}_r(E)$ of Euclidean r-neighbourhoods of edges of G. For the IPGD problem we claim its NP-hardness even if we restrict the graph G to be either a Delaunay triangulation or some of its known subgraphs. We keep the ratio $\mu = \frac{d_{\max}}{d_{\min}}$ bounded from above, thus, imposing an upper bound on the ratio of the largest and smallest aspect ratio of objects from $\mathcal{N}_r(E)$. We show that IPGD remains intractable even in its simple case where $r = \Theta(d_{\max})$ and μ is bounded by some small constant or, equivalently, when objects of $\mathcal{N}_r(E)$ have close constant aspect ratio.

Our first hardness result for IPGD is obtained by using a complexity reduction from the CDC problem. Below we describe a class of hard instances of the CDC problem which correspond to hard instances of the IPGD problem for Delaunay triangulations with relatively small upper bound on the parameter μ.

2.1 NP-hardness of the CDC problem

To single out the class of hard instances of the CDC problem a reduction is used in [9] from the strongly NP-complete minimum dominating set problem which is formulated as follows: given a simple planar graph $G_0 = (V_0, E_0)$ of degree at most 3, find the smallest cardinality set $V_0' \subseteq V_0$ such that for each $u \in V_0 \backslash V_0'$ there is some $v = v(u) \in V_0'$ which is adjacent to u.

Below an integer grid denotes the set of all points on the plane with integer-valued coordinates each of which belongs to some bounded interval. An *orthogonal* drawing of the graph G_0 on some integer grid is the drawing whose vertices are represented by points on that grid whereas its edges are given in the form of polylines that are composed of connected axis-parallel straight line segments of the form $[p_1, p_2]$, $[p_2, p_3], \ldots$, $[p_{k-1}, p_k]$, and intersecting only at the edge endpoints p_1 and p_k, where each point p_i again belongs to the grid. In [9] strong NP-hardness of CDC is proved by reduction from the minimum dominating set problem. This reduction involves using plane orthogonal drawing of G_0 on some integer grid. More specifically, a set D is build on that grid with $V_0 \subset D$. The resulting hard instance of the CDC problem is for the set D and some integer (constant) radius $r_0 \geq 1$. Let us observe that G_0 admits an orthogonal drawing (theorem 1 [15]) on the grid of size $O(|V_0|) \times O(|V_0|)$ whereas total length of each edge is of the order $O(|V_0|)$. Proof of strong NP-hardness of CDC could be conducted taking into account this observation. We can formulate (see theorems 1 and 3 from [9])

Theorem 1. [9] *The* CDC *problem is strongly NP-hard for a constant integer radius r_0 and point sets D on the integer grid of size $O(|D|) \times O(|D|)$. It remains strongly NP-hard even if we restrict centers of radius r_0 disks to be at the points of D.*

Remark 1. For every simple planar graph G_0 of degree at most 3 its orthogonal drawing can be constructed such that at least one its edges is a polyline which is composed of at least two axis-parallel segments.

2.2 NP-hardness of the IPGD problem for Delaunay triangulations

To build a reduction from the CDC problem on the set D (as constructed in proof of the theorem 1 from [9]), we exploit a simple idea that a radius r disk covers a set of points $D' \subset D$ iff a slightly larger disk intersects (and, sometimes, covers) straight line segments, each of which is close to some point of D' and has a small length with respect to distances between points of D. Then a proximity graph H is build whose vertex set coincides with the set of endpoints of small segments corresponding to points of D. Since H usually contains these small segments as its edges, this technique gives NP-hardness for the IPGD problem within numerous classes of proximity graphs. The following technical lemma holds which reports an r-dependent lower bound on the distance between any point with integer coordinates and a radius r circle through the pair of integer-valued points.

Lemma 1. *Let $X \subset \mathbb{Z}^2$, $r \geq 1$ is an integer, $\rho(u; v, w)$ denotes the minimum of two Euclidean distances from an arbitrary point $u \in X$ to the union of two radius r circles which pass through distinct points v and w from X, where $|v-w|_2 \leq 2r$, $\mathbb{Z}$ is the set of integers and $|\cdot|_2$ is Euclidean norm. Then*

$$\min_{u \notin C(v,w),\, v \neq w,\, u,v,w \in X,\, |v-w|_2 \leq 2r} \rho(u; v, w) \geq \frac{1}{480 r^5},$$

where $C(v, w)$ is the union of two radius r circles passing through v and w.

Let us formulate the following restricted form of IPGD.
Vertex Restricted IPGD (VRIPGD(δ)): given a simple plane graph $G = (V, E)$, a constant $\delta > 0$ and a constant $r > 0$, find the least cardinality set $C \subset \mathbb{R}^2$ such that each $e \in E$ is within Euclidean distance r from some point $c = c(e) \in C$ and $C \subset \bigcup_{v \in V} B_\delta(v)$.

Theorem 2. *Both* IPGD *and* VRIPGD(δ) *problems are strongly NP-hard for $r \in [d_{\min}, d_{\max}]$, $\mu = O(n)$ and $\delta = \Theta(r)$ within the class of Delaunay triangulations, where n is the number of vertices in triangulation.*

Proof. Let us prove that IPGD is strongly NP-hard. Proof technique for the VRIPGD(δ) problem is analogous taking into account the theorem 1 (see also proof of the theorem 3 from [9] for details). For any hard instance of the CDC problem, which the theorem 1 reports, the IPGD problem instance is built for $r = r_0 + \delta$ and $\delta = \frac{1}{2000^2 2r_0^{11}}$ as follows. For every $u \in D$ points u_0 and v_0 are found such that $|u - u_0|_\infty \leq \delta/2$ and $|u - v_0|_\infty \leq \delta/2$, where $I_u = [u_0, v_0]$ has Euclidean length at least $\delta/2$ and $|\cdot|_\infty$ denotes norm in $\mathbb{R}^2$ equal to the maximum of absolute values of vector coordinates. More specifically, let us set $I_D = \{I_u = [u_0, v_0] : u \in D\}$. Endpoints of segments from I_D are constructed in sequential manner in polynomial time and space by defining a new segment I_u to provide general position for the set of endpoints of the set $I_{D'} \cup \{I_u\}$, $D' \subset D$, where segments of $I_{D'}$ are already defined. Here endpoints of I_u are chosen in the rational grid that contains u whose elementary cell size is $\frac{c_1}{|D|^2} \times \frac{c_1}{|D|^2}$ for some small absolute rational constant $c_1 = c_1(\delta)$. Assuming $u = (u_x, u_y)$, the point u_0 is chosen in the lower part of the grid with y-coordinates less than $u_y - \delta/4$ whereas v_0 is taken from the upper one for which y-coordinates exceed $u_y + \delta/4$.

Let S be the set of endpoints of segments from I_D. Every disk having I_u as its diameter does not contain any points of S distinct from endpoints of I_u. Let $G = (S, E)$ be a Delaunay triangulation for S which can be computed in polynomial time and space in $|D|$. Obviously, each segment I_u coincides with some edge from E. We have $d_{\min} \leq r$ and $\mu = O(|S|)$. It remains to prove that $r \leq d_{\max}$. Due to the remark 1 and a construction of the set D (see fig. 1 and proof of the theorem 1 from [9]) the set S can be constructed such that the inequality $r \leq d_{\max}$ holds true for G. Moreover, representation length for vertices of S is polynomial with respect to representation length for points of D.

Let k be a positive integer. Obviously, centers of at most k disks of radius r_0, containing D in their union, give centers of radius $r > r_0$ disks whose union is intersected with each segment from E. Conversely, let T be a disk of radius r which intersects a subset $I_{D'} = \{I_u : u \in D'\}$ of segments for some $D' \subseteq D$. When $|D'| = 1$, it is easy to transform T into a disk which contains the segment $I_{D'}$. Points of D have integer coordinates. Moreover, squared Euclidean distance between each pair of points of the subset D' does not exceed $(2r_0 + 4\delta)^2 = 4r_0^2 + 16r_0\delta + 16\delta^2$. As $r_0 \in \mathbb{Z}$, points from D' are located within the distance $2r_0$ from each other. Let us use Helly theorem. Let R be the minimum radius of the disk T_0, containing any triple u_1, u_2 and u_3 from D'. W.l.o.g. we suppose that, say, u_1 and u_2 are on the boundary of T_0 and denote its center by O. Obviously, $R \leq r_0 + 2\delta$. Let us show that the case $R > r_0$ is void. The center of T_0 can be shifted along the midperpendicular to $[u_1, u_2]$ to have u_1 and u_2 at the distance r_0 from the shifted center O'. The distance from the point u_3 to the radius r_0 circle centered at O' does not exceed

$$|O - u_3|_2 + |O - O'|_2 - r_0 \leq 2\delta + \sqrt{(r_0 + 2\delta)^2 - \delta_1^2} - \sqrt{r_0^2 - \delta_1^2} =$$

$$= 2\delta + \frac{4r_0\delta + 4\delta^2}{\sqrt{(r_0 + 2\delta)^2 - \delta_1^2} + \sqrt{r_0^2 - \delta_1^2}} \leq 2\delta + 2\sqrt{r_0\delta + \delta^2} < \frac{1}{480r_0^5},$$

where $\delta_1 = \frac{|u_1 - u_2|_2}{2} \leq r_0$. By the lemma 1 we have $R \leq r_0$. Thus, D' is contained in some disk of radius r_0. Given a set of points on the plane, the smallest radius disk can be found in polynomial time and space which covers this set. Therefore we can convert any set of at most k disks of radius r whose union is intersected with each segment from E to some set of at most k disks of radius r_0 whose union covers D.

Using the corollary 1 of section 4.2 from [2] and the theorem 1 from [11] we arrive at the lower bound $\mu = \Omega\left(\sqrt[3]{n^2}\right)$ which holds true with positive probability for Delaunay triangulations produced by n random uniform points on the unit disk. Thus, the order of the parameter μ for the considered class of hard instances of the IPGD problem is comparable with the one for random Delaunay triangulations.

2.3 NP-hardness of IPGD for other classes of proximity graphs

The same proof technique could be applied for proving NP-hardness of the problem within the other classes of proximity graphs. Let us start with some definitions. The following graphs are connected subgraphs of Delaunay triangulations. A plane graph $G = (S, E)$ is called a *Gabriel graph* when $[u, v] \in E$ iff the disk having $[u, v]$ as its diameter does not contain any other points of S distinct from u and v. A *relative neighbourhood graph* is the plane graph G with the same vertex set for which $[u, v] \in E$ iff there is no any other point $w \in S$ such that $w \neq u, v$ with $\max\{|u - w|_2, |v - w|_2\} < |u - v|_2$. Finally, a plane graph is called a *minimum Euclidean spanning tree* if it is the minimum weight spanning tree of the weighted complete graph $K_{|S|}$ whose vertices are points of S such that its edge weight is given by Euclidean distance between the edge endpoints.

Corollary 1. *Both* IPGD *and* VRIPGD(δ) *problems are strongly NP-hard for* $r \in [d_{\min}, d_{\max}]$, $\mu = O(n)$ *and* $\delta = \Theta(r)$ *within classes of Gabriel, relative neighbourhood graphs and minimum Euclidean spanning trees as well as for* $r \in [d_{\max}, \eta d_{\max}]$ *and* $\mu \leq 4$ *within the class of nearest neighbour graphs where* η *is a large constant.*

3 Positive results

3.1 Polynomial solvability of the IPGD problem for large r

Before presenting polynomially solvable case of the IPGD problem we are to take some preprocessing. It is aimed at reducing the set of points, among which centers of radius r disks are chosen, to a finite set whose cardinality is bounded from above by some polynomial in $|E|$.

Problem preprocessing. As was mentioned in the Introduction, the IPGD problem coincides with the Hitting Set problem considered for Euclidean r-neighbourhoods of graph edges which form the system denoted by $\mathcal{N}_r(E)$. Their boundaries are composed of four parts: two half-circles and two parallel straight line segments. W.l.o.g. we can assume that intersection of any subset of objects from $\mathcal{N}_r(E)$ (if nonempty) contains a point from the intersection of boundaries of two objects from $\mathcal{N}_r(E)$. Thus, our choice of points to form a feasible solution to the IPGD problem can be restricted to the set of intersection points of boundaries of pairs of objects from $\mathcal{N}_r(E)$. The following lemma can be considered a folklore.

Lemma 2. *Let $G = (V, E)$ be a simple plane graph. Each feasible solution C to the* IPGD *problem for G can be converted in polynomial time and space (in $|E|$) to a feasible solution $D \subset D_r(G)$ to* IPGD *for G with $|D| \leq |C|$, where $D_r(G) \subset \mathbb{R}^2$ is some set of cardinality of the order $O(|E|^2)$ which can be constructed in polynomial time and space.*

Polynomially solvable case of IPGD. In distinction to the cases where either $r \in [d_{\min}, d_{\max}]$ or $r = \Theta(d_{\max})$ the IPGD problem is polynomially solvable for $r = \Omega(R(E))$ where $R(E)$ is the smallest radius of the disk that intersects all segments from E. Due to [3] the IPGD problem is solvable in $O(|E|)$ time within the class of plane graphs for which the inequality $r \geq R(E)$ holds uniformly.

Let us consider the IPGD problem within the class of plane graphs for which the inequality $r \geq \eta R(E)$ holds uniformly for some fixed constant $0 < \eta < 1$. Since every radius r disk contains an axis-parallel rectangle whose side is equal to $r\sqrt{2}$, roughly at most $\left\lceil \frac{\sqrt{2}R(E)}{r} \right\rceil^2 \leq \left\lceil \frac{\sqrt{2}}{\eta} \right\rceil^2 = k(\eta) = k$ radius r disks are needed to intersect all segments from E. Therefore the brute-force search algorithm could be applied that just sequentially tries each subset of $D_r(G)$ of cardinality at most k. This amounts roughly to $O\left(k^2|E|^{2k+1}\right)$ time complexity. Thus, we arrive at the polynomial time algorithm whose complexity depends exponentially on $1/\eta$. This algorithm gives an optimal solution to the IPGD problem taking the lemma 2 into account.

3.2 Approximation algorithm for the IPGD problem

Below the approximation algorithm is reported for the IPGD problem whose approximation factor depends on the maximum aspect ratio among objects of

$\mathcal{N}_r(E)$. More specifically, let us focus on the case of IPGD where the inequality $r \geq \frac{d_{\max}}{2\lambda}$ holds uniformly within some class $\mathcal{G}_\lambda$ of simple plane graphs for a constant $\lambda > 0$. It corresponds to the situation where objects from the system $\mathcal{N}_r(E)$ have their aspect ratio bounded from above by $1+\lambda$. In this case it turns out that the problem admits an $O(1)$-approximation algorithm whose factor depends on λ. The following auxiliary problem is considered to formulate it.

Cover endpoints of segments with disks (CESD). Let $S(G) \subseteq V$ be the set of endpoints of edges of G. It is required to find the smallest cardinality set of radius r disks whose union contains $S(G)$.

Algorithm. Compute and output 8-approximate solution to the CESD problem using $O(|E| \log OPT_{CESD}(S(G), r))$-time algorithm (see sections 2 and 4 from [8]).

We call a subset $V' \subseteq V$ by a *vertex cover* for $G = (V, E)$ when $e \cap V' \neq \varnothing$ for any $e \in E$. The statement below bounds the ratio of optima for CESD and IPGD problems in the general case where $S(G)$ is an arbitrary vertex cover of the graph G.

Statement 3 *The following bound holds true for any graph $G \in \mathcal{G}_\lambda$ without isolated vertices:*

$$\frac{OPT_{CESD}(S(G), r)}{OPT_{IPGD}(G, r)} \leq p(1 + 2\lambda)$$

where $p(x)$ is the smallest number of unit disks needed to cover radius x disk.

Proof. Let $C_0 = C_0(G, r) \subset \mathbb{R}^2$ be an optimal solution to IPGD for a given $G \in \mathcal{G}_\lambda$. Set $E(c, G) := \{e \in E\colon c \in N_r(e)\}$, $c \in C_0$. For every $e \in E(c, G)$ there is a point $c(e) \in e$ with $|c - c(e)|_2 \leq r$. Any point from the set $S(c, G)$ of endpoints of segments from $E(c, G)$ is within the distance $r + d_{\max}$ from the point c. Due to definition of p, at most $p(1 + 2\lambda)$ radius r disks are needed to cover radius $r + d_{\max}$ disk. Therefore the set $S(G) \subseteq \bigcup\limits_{c \in C_0} S(c, G)$ is contained in the union of at most $|C_0| p(1 + 2\lambda)$ radius r disks.

Corollary 2. *The algorithm is $8p(1 + 2\lambda)$-approximate.*

Remark 2. Approximation factor of the algorithm is in fact lower when $\mathcal{G}_\lambda$ is the subclass of Delaunay triangulations or of their subgraphs. Indeed, in this case there is no need to cover the whole radius $r + d_{\max}$ disk with radius r disks.

Remark 3. If $S(G)$ is the set of midpoints of segments from E, the algorithm is $8p(1 + \lambda)$-approximate.

4 Conclusion

Complexity and approximability are studied for the problem of intersecting a structured set of straight line segments with the smallest number of disks of radii $r > 0$ where a structural information about segments is given in the form of an edge set of a plane graph. It is shown that the problem is strongly NP-hard within the class of Delaunay triangulations and some of their subgraphs for small and medium values of r while for large r it is polynomially solvable. Fast approximation algorithm is given for the IPGD problem whose approximation factor depends on the maximum aspect ratio among objects from $\mathcal{N}_r(E)$. Of course, those algorithms are of particular interest whose factor is bounded from above by some absolute constant. This sort of algorithms is our special focus for future research.

References

1. P. K. Agarwal, A. Efrat, S. K. Ganjugunte, D. Hay, S. Sankararaman, G. Zussman. The resilience of WDM networks to probabilistic geographical failures. *IEEE/ACM Trans. Netw.*, 21(5):1525–1538, 2013.
2. E. M. Arkin, A. F. Anta, J. S. Mitchell, M. A. Mosteiro. Probabilistic bounds on the length of a longest edge in Delaunay graphs of random points in d dimensions. *Comput. Geom.*, 48(2):134–146, 2015.
3. B. K. Bhattacharya, S. Jadhav, A. Mukhopadhyay, J. M. Robert. Optimal algorithms for some intersection radius problems. *Computing*, 52(3):269–279, 1994.
4. P. Bose, J.L. De Carufel, S. Durocher, P. Taslakian. Competitive online routing on Delaunay triangulations. *Lecture Notes in Computer Science*, 8503:98–109, 2014.
5. P. Bose, D. G. Kirkpatrick, Z. Li. Worst-case-optimal algorithms for guarding planar graphs and polyhedral surfaces. *Comput. Geom.*, 26(3):209–219, 2003.
6. T. M. Chan, E. Grant. Exact algorithms and APX-hardness results for geometric packing and covering problems. *Comput. Geom.*, 47(2):112–124, 2014.
7. A. Efrat, M.J. Katz, F. Nielsen, M. Sharir. Dynamic data structures for fat objects and their applications. *Comput. Geom.*, 15:215–227, 2000.
8. T. F. Gonzalez. Covering a set of points in multidimensional space. *Inf. Process. Lett.*, 40(4):181–188, 1991.
9. T. Hasegawa, S. Masuyama, T. Ibaraki. Computational complexity of the m-center problems on the plane. *Transactions of the Institute of Electronics and Communication Engineers of Japan. Section E*, E64(2):57–64, 1981.
10. D. Marx. Efficient approximation schemes for geometric problems? *Lecture Notes in Computer Science*, 3669:448–459, 2005.
11. T. Onoyama, M. Sibuya, H. Tanaka. Limit Distribution of the Minimum Distance between Independent and Identically Distributed d-Dimensional Random Variables. *Statistical Extremes and Applications*, 549–562, 1984.
12. J. O'Rourke. *Art Gallery Theorems and Algorithms.* Oxford University Press, 1987.
13. K. Quanrud, S. Har-Peled. Approximation algorithms for polynomial-expansion and low-density graphs. *Lecture Notes in Computer Science*, 9294:717–728, 2015.
14. I. Stojmenovic, J. Urrutia, P. Bose, P. Morin. Routing with guaranteed delivery in ad hoc wireless networks. *Wireless Networks*, 7(6):609–616, 2001.
15. R. Tamassia, I.G. Tollis. Planar grid embedding in linear time. *IEEE Transactions on Circuits and Systems*, 36:1230–1234, 1989.

A Proof of the Lemma 1

Proof. Let $u = (x, y)$, $v = (x_1, y_1)$ and $w = (x_2, y_2)$ be distinct points of X. Consider an arbitrary radius r circle (out of two circles) which passes through v and w, and denote its center by O. A lower bound is obtained below for the distance $\pi = \pi(u; v, w)$ from that circle to the point $u \notin C(v, w)$.

Let $\varDelta = |v-w|_2$, $\lambda = \sqrt{r^2 - \frac{\varDelta^2}{4}}$, $a = (u-v, u-w)$ and $b = (u-v, (v-w)^\perp)$, where $(v-w)^\perp = \pm(y_1 - y_2, -x_1 + x_2)$. The distance $\pi > 0$ can be written in the form:

$$\pi = \pi(u; v, w) = \left| \left| \frac{v+w}{2} - \lambda \frac{(v-w)^\perp}{|v-w|_2} - u \right|_2 - r \right| = \left| \frac{a + \frac{2\lambda b}{\varDelta}}{\sqrt{a + \frac{2\lambda b}{\varDelta} + r^2} + r} \right|.$$

Without loss of generality it assumed that u is in the $2r$ radius disk centered at O. Indeed, otherwise $\pi \geq r \geq \frac{1}{r}$. Let us bound denominator of fraction π, taking into account that $\varDelta \leq 2r$, $|u-v|_2 \leq |u-O|_2 + |O-v|_2 \leq 3r$ and $|b|/\varDelta \leq 3r$:

$$\sqrt{a + \frac{2\lambda b}{\varDelta} + r^2} + r \leq 5r.$$

As points of X have integer coordinates, a and b are integers. For $\varDelta^2 = 4r^2$ we get $\pi \geq \frac{1}{5r}$. When $\varDelta^2 \leq 4r^2 - 1$ it is enough to prove the inequality

$$\left| a + \frac{2\lambda b}{\varDelta} \right| \geq \frac{1}{96r^4}.$$

Indeed, again, combining this bound with the aforementioned upper bound for denominator of the fraction π, we get $\pi \geq \frac{1}{480r^5}$.

For integer $\frac{2\lambda b}{\varDelta}$ the left-hand side of the inequality is at least 1. Thus, it remains for us to prove the inequality for the case where $\frac{2\lambda b}{\varDelta} \notin \mathbb{Z}$. Suppose that $q = \left\{ \left| \frac{2\lambda b}{\varDelta} \right| \right\} > 0$ and $k = \left[\left| \frac{2\lambda b}{\varDelta} \right| \right]$, where $\{\cdot\}$ and $[\cdot]$ denote fractional and integer part of real number respectively. In fact, the term $\min\{q, 1-q\}$ is bounded from below. Let us start estimating with q. First, it is assumed that $\gamma = \frac{4r^2b^2}{\varDelta^2} \in \mathbb{Z}$. We have $k^2 < \frac{4\lambda^2 b^2}{\varDelta^2} < (k+1)^2$. As $q > 0$, we get $q \geq \left\{ \sqrt{k^2+1} \right\}$. Due to concavity of the square root we have

$$\left\{ \sqrt{k^2+1} \right\} = \left\{ \sqrt{\frac{2k \cdot k^2}{2k+1} + \frac{(k+1)^2}{2k+1}} \right\} \geq \left\{ k + \frac{1}{2k+1} \right\} = \frac{1}{2k+1} \geq$$

$$\geq \frac{1}{\frac{4\lambda |b|}{\varDelta} + 1} \geq \frac{1}{13r^2}.$$

Now the case is considered where $\gamma \notin \mathbb{Z}$. As $2kq + q^2 \geq \{2kq + q^2\} = \{\gamma\}$, we have that

$$q \geq \sqrt{k^2 + \{\gamma\}} - k \geq \frac{\{\gamma\}}{\sqrt{k^2 + \{\gamma\}} + k} \geq \frac{\frac{1}{\varDelta^2}}{\frac{4r|b|}{\varDelta}} \geq \frac{1}{12r^2\varDelta^2} \geq \frac{1}{48r^4}.$$

Let us get a lower bound for $1-q$. Again, assume that $\gamma\in\mathbb{Z}$. Arguing analogously, we arrive at the bound

$$2k(1-q)+(1-q)^2\geq\{(k+1-q)^2\}=\left\{(k+1)^2-\frac{4\lambda^2b^2}{\Delta^2}-2q(1-q)\right\}\geq\frac{1}{2}.$$

Resolving the quadratic inequality with respect to $1-q$, we get:

$$1-q\geq\sqrt{k^2+\frac{1}{2}}-k=\frac{\frac{1}{2}}{\sqrt{k^2+\frac{1}{2}}+k}\geq\frac{1}{\frac{8r|b|}{\Delta}}\geq\frac{1}{24r^2}.$$

Now let $\gamma\notin\mathbb{Z}$. Let us consider the subcase where $\{\gamma\}+2q(1-q)>1$. We get

$$\left\{(k+1)^2-\frac{4\lambda^2b^2}{\Delta^2}-2q(1-q)\right\}\geq 1-\{\gamma\}\geq\frac{1}{\Delta^2}.$$

Resolving the corresponding inequality with respect to $1-q$, we arrive at the analogous lower bound $1-q\geq\frac{1}{48r^4}$.

Now we are to address the case where $\{\gamma\}+2q(1-q)<1$. Obviously

$$\left\{(k+1)^2-\frac{4\lambda^2b^2}{\Delta^2}-2q(1-q)\right\}=1-\{\gamma\}-2q(1-q).$$

For $1-q<\frac{1}{4\Delta^2}$ we have $1-\{\gamma\}-2q(1-q)\geq\frac{1}{\Delta^2}-\frac{1}{2\Delta^2}=\frac{1}{2\Delta^2}$. Arguing analogously, we obtain the following bound $1-q\geq\frac{1}{96r^4}$; otherwise $1-q\geq\frac{1}{4\Delta^2}\geq\frac{1}{16r^2}$.

For $\{\gamma\}+2q(1-q)=1$ we have:

$$1-q=\frac{1-\{\gamma\}}{2q}\geq\frac{1-\{\gamma\}}{2}\geq\frac{1}{2\Delta^2}\geq\frac{1}{8r^2}.$$

Finally, we get the claimed bound $\min\{q,1-q\}\geq\frac{1}{96r^4}$.